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SUPPORT AND SEMINORM INTEGRABILITY THEOREMS FOR R-SEMISTABLE PR--ETC(U)

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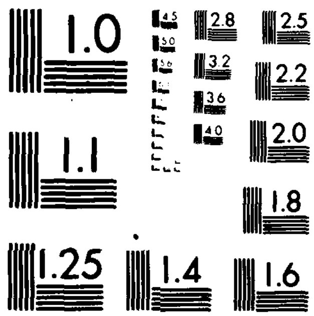

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ABSTRACT

Let μ be an r-semistable K-regular probability measure of index $\alpha \in (0, 2]$ on a complete locally convex topological vector space E. It is shown that the topological support S_μ of μ is a translated convex cone if $\alpha \in (0, 1)$, and a translated truncated cone if $\alpha \in (1, 2]$. Further, if $\alpha = 1$ and μ is symmetric, then it is shown that S_μ is a vector subspace of E. These results subsume all earlier known results regarding the support of stable measures. A result regarding the support of infinitely divisible probability measure on E is also obtained. A seminorm integrability theorem is obtained for K-regular r-semistable probability measures μ on E. The result of de Acosta (Ann. of Probability, 3(1975), 865 - 875)and Kanter (Trans. Seventh Prague Conf., (1974), 317 - 323) is included in this theorem as long as the measures are defined on LCTVS and seminorm is continuous.

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1. INTRODUCTION

Let E be a complete locally convex topological vector space (LCTVS) and let μ be a stable probability measure (p.m.) of index $\alpha \in (0, 2]$; then it is shown by Tortrat [15] that for $\alpha \neq 1$, S_μ, the support of μ, is a certain cone (if μ is symmetric, then it is shown by Rajput [13, 14] that S_μ is a subspace for all α; this result for $1 < \alpha \leq 2$ is also obtained by de Acosta [1]). Furthermore, if p is a continuous seminorm (in fact measurability is enough) on E, then it is shown by de Acosta [1] and Kantor [8] that

$$\int_E p^\delta(x)\ \mu(dx) < \infty, \text{ for all } 0 \leq \delta < \alpha .$$

A natural and nontrivial generalization of stable measures is the class of r-semistable measures, which was first introduced and studied on the real line R by Paul Lévy [12]. Later, Kruglov, in an interesting paper [9], obtained a quite explicit form of the characteristic function of r-semistable p. measures on R and showed that this class has properties similar to those of stable p. measures (similar situation is true in Hilbert space is shown by Kruglov [10] and by Kumar [11]). Partialy motivated from these papers, we raised and completely answered, in this paper, the question of whether r-semistable p. measures have properties similar to those of stable p. measures mentioned above. Explicitly, we obtain the following results: Let μ be a K-regular r-semistable p. measure (see Definition 2.1) of index $\alpha \in (0, 2]$ on a complete LCTVS E, then S_μ, the support of μ, is a translated convex cone or a translated truncated cone according as whether $0 < \alpha < 1$ or $1 < \alpha \leq 2$; further, if $\alpha = 1$ and μ is symmetric, we prove that S_μ is a subspace (Theorem 3.2). This result subsumes all earlier known results regarding the support of stable measures [1, 4, 13, 14, 15]. (A general theorem which gives a formula for the support of K-regular infinitely divisible (i.d.) p. measures on E and which includes some

results for the supports of i.d. measures derived in [4, 14, 15] is also obtained). Let μ and E be as above and p a continuous seminorm on E ; then $\int_E p^\delta(x)\,\mu(dx) < \infty$, if $0 \leq \delta < \alpha$. This result includes the seminorm integrability theorem for stable measures in [1, 8], as long as the measures are defined on LCTVS and p is continuous.

Our proof of the support theorem for i.d. measures uses similar ideas to those of Brockett [4], who proved part of our result in Hilbert spaces, and Tortrat [15, 16], who proved similar results under different hypotheses in certain LC spaces. Our techniques of proof of the support theorem for r-semistable measures, however, seem new and quite interesting. Our proof of the seminorm integrability result is classical and has the drawback in that it uses a strong central limit theorem in Banach spaces [2].

2. PRELIMINARIES

Unless otherwise stated, the following conventions and notation will remain fixed in this paper:

All vector spaces considered are over the real field R and all topological spaces are assumed Handsdorff. If μ and ν are two finite K-regular p. measures on the Borel σ-algebra $\mathcal{B}$ of a topological vector space E , then μ^{*n} and $\mu * \nu$ will denote, respectively, μ convoluted n-times and the convolution of μ and ν . If $a \neq 0$, then T_a will denote the map on E defined by $T_a(x) = ax$, $x \in E$; further $T_a\mu$ will denote the measure $\mu \circ T_a^{-1}$. For any $x \in E$, δ_x will denote the degenerate measure at x . E and E^* will, respectively, denote a complete LCTVS and its topological dual, and $M_K(E)$ will denote the class of all K-regular p. measures on E . If A is a subset of a topological space, then $\bar{A}$ will denote its closure; finally, θ will denote the zero element of E .

We will now give the definition of r-semistable p. measures and some of their properties pertinent to this paper. This definition and results are taken from Chung, Rajput and Tortrat [5], which may be referred to for

other properties of r-semistable p. measures. The first result below dealing with i.d. p.m. is taken from [6, 7] .

<u>Definition 2.1</u>: Let E be a LCTVS , $\mu \in M_K(E)$ and $0 < r \leq 1$. Then μ is said to be <u>r-semistable</u> if there exists a K-regular p. measure ν , sequences $\{a_n\} \subseteq R$, $a_n > 0$, and $\{x_n\} \subseteq E$, and an increasing sequence of positive integers $\{k_n\}$ such that

$$\frac{k_n}{k_{n+1}} \longrightarrow r$$

and

$$T_{a_n} \nu^{*k_n} * \delta_{x_n} \xrightarrow{w} \mu \ ,$$

as $n \longrightarrow \infty$ (the symbol ' $\xrightarrow{w}$ ' will always denote the weak convergence).

(i) Let $\mu \in M_K(E)$ be i.d. then there exists a measure F (called the Lévy measure), a quadratic form Q on E^* , an $x_0 \in E$, and a compact convex circled subset K of E with $F(K^c) < \infty$ such that, for every $f \in E^*$, the characteristic function $\hat{\mu}$ of μ has the representation

$$\hat{\mu}(f) = \exp\{if(x_0) - \tfrac{1}{2}Q(f) + \int_E \psi(f, x)dF(x)\} \ ,$$

where $\psi(f, x) = e^{if(x)} - 1 - if(x)\, I_K(x)$ (I_K is the indicator of K) ; further, Q and F are unique and x_0 depends on the choice of K . For the sake of simplicity of notation we will use the notation $[x_0, Q, K, F]$ to denote the above representation for μ .

(ii) Let μ be as above with the representation $[x_0, Q, K, F]$, then there exists a unique continuous (in weak topology) semigroup $\{\mu^s : s > 0\}$ of K-regular i.d. p. measures with $\mu = \mu^1$ (μ^s is referred to as the s^{th} root of μ and has the representation $[s\, x_0, s\, Q, K, sF]$), and

$$(\mu^s)^t = \mu^{st}. \tag{2.1}$$

(iii) Let $\mu \in M_K(E)$ and $r \in (0, 1)$, then μ is r-semistable if and only if μ is i.d. and there exist a unique $\alpha \in (0, 2)$ and $x(r_n) \in E$ such that

$$\mu^{r^n} = T_{r^{n/\alpha}} \mu * \delta_{x(r_n)}, \tag{2.2}$$

for all $n = 1, 2, \ldots$. The number α is referred to as the index of μ ($\alpha = 2$ corresponds to the Gaussian case).

(iv) Let $\mu \in M_K(E)$ then μ is 1-semistable $\Leftrightarrow \mu$ is r-semistable for every $r \in (0, 1) \Leftrightarrow \mu$ is stable.

(v) The class of stable K-regular p. measures are properly contained in the class of r-semistable p. measures for every fixed $r \in (0, 1)$.

3. SUPPORT THEOREMS FOR I.D. AND r-SEMISTABLE PROBABILITY MEASURES

We recall that the support of a finite Borel measure μ on a topological space is, by definition, the smallest closed set (if it exists) with full μ-measure. If μ is K-regular (or even τ-regular) the support of μ always exists. The main purpose of this section is to prove the following two theorems.

Theorem 3.1: Let μ be an i.d. K-regular p.m. on E with representation $[\theta, 0, K, F]$:

(i) Let $\mathcal{U}$ be the class of all convex circled Borel nbds. of θ directed by reverse set inclusion; set $F_0 = F/K^c$, $F_U = F/K \cap U^c$, $a_U = \int_E x dF_U(x)$, $\nu_0 = e(F_0)$, and $\nu_U = e(F_U)$ (note $a_U \in E$, see [7]), then

$$S_\mu = [\bigcap_V \{\overline{\bigcup_{U \geq V} (S_{\nu_U} + a_U)}\} + S_{\nu_0}]^{-} . \tag{3.1}$$

In addition if $\{\delta_{a_U}\}$ is tight and δ_a is any limit pt. of $\{\delta_{a_U}\}$, then

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K^c and U^c , respectively, denote the complements of K and U .

$$S_\mu = a + \overline{G(F)} \ ,$$

where $G(F)$ is the semigroup with zero element which is generated by S_F, the support of F ($S_F = \{x \in E: F(V_x) > 0$, for every open nbd. V_x of $x\}$).

(ii) (Tortrat) If $\int_K p_K(x)\, dF(x) < \infty$, where p_K is the Minkowski functional of K which is assumed to take the value $+\infty$ off the set $\bigcup_{n=1}^{\infty} nK$, then $\{\delta_{a_U}\}$ is tight (a_U is as in (i)); hence $S_\mu = a + \overline{G(F)}$, where δ_a is any limit point of $\{\delta_{a_U}\}$.

(iii) If $\int_K p_K^2(x)\, dF(x) < \infty$, then $S_\mu = \overline{G(F) + A}$, where A is a closed set.

Theorem 3.2: Let μ be a K-regular r-semistable p.m., $r \in (0, 1)$, of index $\alpha \in (0, 2]$ on E.

(i) If $\alpha \in (1, 2]$, then S_μ is a translate of a truncated cone; further, if μ is strictly r-semistable (i.e. $x(r) = \theta$ in (2.2)), then S_μ is a truncated cone.

(ii) If $\alpha \in (0, 1)$, then S_μ is a translate of a convex cone; further, if μ is strictly r-semistable, then S_μ is a convex cone.

(iii) If $\alpha = 1$ and μ is symmetric, then S_μ is a subspace.

Remark 3.3: As hinted in Section 1, part (iii) of Theorem 3.1 and the fact that $S_\mu = a + \overline{G(F)}$ under a hypothesis similar to $\int_K p_K(x)dF(x) < \infty$, was obtained, in the Hilbert space setting, by Brockett [4] and the last statement, *under certain other hypotheses*, was obtained, in LCTV setting, by Tortrat [15, 16]. Our proof of Theorem 3.1 uses similar ideas as those of [4]; however, because of the weaker structure available in arbitrary LCTV spaces, modifications of techniques are required. Since clearly, from Definition 2.1, *every stable measure is r-semistable for all* r, Theorem 3.2 includes the support results regarding stable measures obtained in [1, 4, 13, 14, 15] ; and, in view of Section 2, the above theorem also provides the corresponding results for 1-semistable measures.

For the proof of Theorems 3.1 and 3.2, we will need the following lemmas. The proof of Lemma 3.4 is elementary and Lemma 3.5 is well known. Lemma 3.6 was first conceived in [17] in the locally compact group setting; the proof presented here is similar to the one in [17], but certain details need to be verified. The last Lemma is taken from [5].

<u>Lemma 3.4</u>: Let $r \in (0, 1)$ and $\alpha \geq 1$. Set $A = \{r^{m/\alpha}k: k = 1, 2, \ldots, [1/r^m], m = 1, 2, \ldots\}$, where $[x]$ denotes the integral part of the number x. Then A is dense in $[0, \infty)$ if $\alpha > 1$, and A is dense in $[0, 1]$ if $\alpha = 1$.

<u>Lemma 3.5</u>: Let μ and ν be two K-regular p. measures on a LCTVS E and $a \in R$, $a \neq 0$. Then

$$S_{T_a\mu} = aS_\mu \quad \text{and} \quad S_{\mu*\nu} = \overline{[S_\mu + S_\nu]} .$$

<u>Lemma 3.6</u>: Let $\{\nu_n\}$ and $\{\lambda_n\}$ be two nets of K-regular p. measures on a LCTVS E and let ν be a K-regular p.m. on E. Assume $\nu = \nu_n * \lambda_n$, for each n, $\{\nu_n\}$ is tight, and $\nu_n \xrightarrow{w} \nu$. Then $\lambda_n \longrightarrow \delta_\theta$ and $S_\nu = \bigcap_m \overline{[\bigcup_{n\geq m} S_{\nu_n}]}$. Further, if $S_{\nu_n} \uparrow$ with n, then $S_\nu = \overline{[\bigcup_n S_{\nu_n}]}$.

<u>Proof</u>: From [6], $\{\lambda_n\}$ is tight; hence it has a subnet which converges to a K-regular p.m. λ. This implies $\nu = \nu * \lambda$. Hence (using characteristic functions) $\lambda = \delta_\theta$. Now, by repeating the above argument replacing $\{\lambda_n\}$ by any subnet of it, we have that each subnet of $\{\lambda_n\}$ in turn has a subnet converging to δ_θ. This shows $\lambda_n \xrightarrow{w} \delta_\theta$.

Now we prove the second part. For each fixed m, let $U_m = E \setminus \overline{\bigcup_{n>m} S_{\nu_n}}$. Then $\nu_n(U_m) = 0$ (by the definition of the support), for all $n \geq m$. But, since $\nu_n \xrightarrow{w} \nu$, $\liminf_n \nu_n(U_m) \geq \nu(U_m)$. This implies $\nu(U_m) = 0$, for every m. So $S_\nu \subseteq \bigcap_m \overline{[\bigcup_{n\geq m} S_{\nu_n}]}$. To prove the reverse inclusion, let $x \in \bigcap_m \overline{[\bigcup_{n\geq m} S_{\nu_n}]}$ and U be an arbitrary open nbd. of θ. It follows

that there exists a subnet $\{m_k\}$ of $\{m\}$ such that $(x + W) \cap S_{\nu_{m_k}} \neq \emptyset$, where W is a closed nbd. of θ such that $W + W \subseteq U$. Then $W \subseteq U - y$, for every $y \in W$. From this and $\nu = \nu_{m_k} * \lambda_{m_k}$, we have

$$\nu(x + U) = \int_E \nu_{m_k} (U - y + x)\, \lambda_{m_k}(dy) \geq \nu_{m_k}(W + x)\, \lambda_{m_k}(W) ,$$

for all k. Taking k large and noting that $\lambda_{m_k} \xrightarrow{W} \delta_\theta$ and $\nu_{m_k}(W + x) > 0$, for all k (as shown above), we have $\nu(x + U) > 0$. This shows $x \in S_\nu$, which completes the proof of the second part. The proof of the last part is now obvious.

Note that in the above the hypothesis of tightness on $\{\nu_n\}$ is needed only to conclude $\lambda_n \xrightarrow{W} \delta_\theta$. Thus if $\lambda_n \xrightarrow{W} \delta_\theta$ were already in the hypothesis of the lemma, then the conclusions would hold without the tightness hypothesis on $\{\nu_n\}$. This observation will be used in the proofs of Theorems 3.1 and 3.2.

<u>Lemma 3.7</u>: Let μ be a K-regular strictly r-semistable p.m. of index $\alpha \in (0, 1)$ on E. Then $\hat{\mu}(f) = \exp\{\int_E (e^{if(x)} - 1)dF(x)\}$ and $\int_K p_K(x)\, dF(x) < \infty$, where F is the Lévy measure of μ and K is the compact convex circuled set appearing in the Lévy representation of μ (note μ is i.d.).

We are now ready to prove Theorems 3.1 and 3.2.

<u>Proof of Theorem 3.1 (i)</u>: It is shown in [7] that $\nu_U * \delta_{a_U} \xrightarrow{W} \mu_0$, with $\hat{\mu}_0(f) = \exp\{\int_K (e^{if(x)} - 1 - i\, f(x))dF(x)\}$, that $\mu_0 = \nu_U * \delta_{a_U} * \lambda_U$, with λ_U i.d. and K-regular, for every $U \in \mathcal{U}$, and that $\lambda_U \xrightarrow{W} \delta_\theta$ (note $\mu = [\theta, 0, K, F]$). Lemma 3.6 applies and we get $S_{\mu_0} = \bigcap_V [\bigcup_{U \geq V} (S_{\nu_U} + a_U)]^{-}$. Then, since $\mu = \mu_0 * \nu_0$, we get (3.1). To prove the second part denote by δ_a the limit of a subnet of $\{\delta_{a_U}\}$ and use the same notation for the subnet. Then $\mu_0 * \delta_{-a} = \nu_U * \lambda_U * \delta_{a_U - a}$, $\lambda_U * \delta_{a_U - a} \xrightarrow{W} \delta_\theta$ and $\nu_U \xrightarrow{W} \mu_0 * \delta_{-a}$. Thus, since $S_{\nu_U} \downarrow$ with U,

we have, from Lemma 3.6, $S_{\mu_0} = \overline{\bigcup S_{\nu_U}} + a$. Therefore, $S_\mu = a + \overline{\bigcup S_{\nu_U} + S_{\nu_0}}$. But (see, for example, [14]), $\overline{\bigcup S_{\nu_U} + S_{\nu_0}} = \overline{G(F)}$, we have $S_\mu = a + \overline{G(F)}$.

Proof of Theorem 3.1 (ii): For any $f \in E^*$, we have

$$|f(a_U)| = |\int_{K\cap U^c} f(x) d F(x)| \le p^+_{K^0}(f) \int_K p_K(x)\, d(F(x)\ ,$$

$$\le \text{ const. } p_{K^0}(f)\ ;$$

hence, by the Bipolar theorem, $\{a_U\}$ is contained in a compact subset. Showing $\{\delta_{a_U}\}$ is tight.

Proof of Theorem 3.1 (iii): Denote by M the measure which is equal to F on K and 0 off K and recall that $\mu = \mu_0 * \nu_0$ (see the proof of (i)) . The condition $\int_K p_K^2(x)\, dF(x) < \infty$ implies

$$\mu_0(f) = e^{if(a_0)} \exp\{\int_E (e^{if(x)} - 1 - \frac{if(x)}{1 + p_K^2(x)})\, dM(x)\},\ \text{for some } a_0 \in E,$$

(see [6]). Now define, for every $U \in \mathcal{U}$ ($\mathcal{U}$ is as in (i)) , $M_U = M$ on $(K \cap U)^c$ and $M_U(B) = \int_B p_K(x)\, dM(x)$, if B is a Borel subset of $K \cap U$. Clearly M_U is equivalent to M and, since $M_U \le M$, M_U is a Lévy measure [6]. Denote by α_U the K-regular i.d. p.m. with ch. function

$$\hat{\alpha}_U(f) = \exp\{\int_E (e^{if(x)} - 1 - \frac{if(x)}{1 + p_K^2(x)})\, dM_U(x)\}\ ;$$

it follows that $\mu_0 * \delta_{-a_0} = \alpha_U * \beta_U$, for some K-regular i.d. p.m. β_U , for every $U \in \mathcal{U}$. Now, since for $f \in E^*$,

$$|\int_E [\frac{f(x)}{1 + p_K^2(x)}]\, dM_U(x)|$$

$$\le |\int_{(K\cap U)^c} [\frac{f(x)}{1 + p_K^2(x)}] dM(x)| + |\int_{K\cap U} [\frac{p_K(x)\ f(x)}{1 + p_K^2(x)}]\, dF(x)|$$

+ p_{K^0} denotes the Minkowski functional of K^0 , the polar of K .

$$= \int_{K\cap U^c} \frac{f(x)}{1 + p_k^2(x)} \, dF(x)| + |\int_{K\cap U} \frac{p_K(x)f(x)}{1 + p_k^2(x)} \, dF(x)|$$

$$\leq p_{K^\circ}(f) \; [F(U^c) + \int_K p_k^2(x) \, dF(x)] \; ;$$

it follows that $b_U = \int_E [\frac{x}{1 + p_K^2(x)}] dM_U(x)$ belongs to E and $\hat{\alpha}_U(f) = e^{if(b_U)} \exp\{ \int_E (e^{if(x)} - 1) \, dM_U(x)\}$. Therefore, since $\int_{K\cap U} p_K(x) \, dM_U(x) = \int_K p_K^2(x) \, dF(x) < \infty$, using what we have proved in (ii) and replacing K by $K \cap U$ (with U a closed nbd. of θ), we have, for some $b_U' \in E$, $S_{\alpha_U} = b_U' + \overline{G(M_U)} = b_U' + \overline{G(M)}$, since M is equivalent to M_U .

Hence

$$S_\mu = \overline{[S_{\mu_0} + S_{\nu_0}]} = a_0 + \overline{[S_{\alpha_U} + S_{\beta_U} + S_{\nu_0}]}$$

(for a fixed closed nbd. U of θ) ,

$$= a_0 + b_U' + \overline{[G(M) + S_{\nu_0} + S_{\beta_U}]}$$

$$= \overline{G(F) + A} \, ,$$

where $A = S_{\beta_U} + a_0 + b_U'$ (note $\overline{G(F)} = \overline{[G(M) + S_{\nu_0}]}$) .

This completes the proof of Theorem 3.1.

<u>Proof of Theorem 3.2 (i)</u>: According to [5], μ can be centered, i.e., there exists an $x_0 \in E$ and a strictly r-semistable p.m. ν with the same index such that $\mu = \nu * \delta_{x_0}$. Thus, to complete the proof of (i), we need to show that S_ν is a truncated cone. We first show that $sS_\nu \subseteq S_\nu$, for any $s \geq 1$. Let $s \geq 1$ and set $t = s - 1 \geq 0$. Using Lemma 3.4, we

choose a sequence $\{k_n\}$ of positive integers such that $1 \leq k_n \leq [1/r^n]$ and $t_n \equiv r^{n/\alpha} k_n \to t$, as $n \to \infty$. Then, since $r^{n(1 - 1/\alpha)} \to 0$ (note $1 < \alpha$), as $n \to \infty$, and $r^{n(1 - 1/\alpha)} r^{n/\alpha} k_n = r^n k_n$, we have $r^n k_n \to 0$, as $n \to \infty$. Therefore, by semigroup and continuity property of $\{\mu^p: p > 0\}$ (see Section 2), we have

$$\mu^{r^n k_n} * \mu^{1-r^n k_n} = \mu$$

and

$$\mu_n \equiv \mu^{1-r^n k_n} \xrightarrow{w} \mu ,$$

as $n \to \infty$. Therefore, using the fact that $\mu^{r^n k_n} = (\mu^{r^n})^{*k_n} = T_{r^{n/\alpha}} \mu^{*k_n}$, it follows, from Lemmas 3.5 and 3.6 (note that $\{\mu^p: 0 < p \leq p_0\}$ is tight (see Section 2)), that

$$S_\mu = [\overline{r^{n/\alpha} S_\mu^{(k_n)} + S_{\mu_n}}] , \tag{3.2}$$

for each $n = 1, 2, \ldots$, and

$$S_\mu = \bigcap_{j=1}^{\infty} [\overline{\bigcup_{n \geq j} S_{\mu_n}}] , \tag{3.3}$$

where $S_\mu^{(k_n)}$ denotes the k_n-fold sum of S_μ. Now let $x \in S_\mu$. Then, by (3.3), for each $j = 1, 2, \ldots$,

$$x \in [\overline{\bigcup_{n \geq j} S_{\mu_n}}] . \tag{3.4}$$

Let $\mathcal{D}$ be the set of pairs (W, n), where W is an open nbd. of x and n is a positive integer such that $W \cap S_{\mu_n} \neq \phi$. Define the relation

$\leq$ on $\mathcal{D}$ by $(W_1, n_1) \leq (W_2, n_2)$ if and only if $W_2 \subseteq W_1$ and $n_1 \leq n_2$. Using (3.4), we can easily verify that $(\mathcal{D}, \leq)$ is a directed set. Let $x_{(W,\, n)}$ be any element in $W \cap S_{\mu_n}$ and let $t_{(W,\, n)} = t_n$. Then $\{t_{(W,\, n)}\}$ is a subnet of $\{t_n\}$ and $x_{(W,\, n)} \to x$. Now, by (3.2), $t_{(W,\, n)}x + x_{(W,\, n)} \in S_\mu$; and, clearly, $t_{(W,\, n)}x + x_{(W,\, n)} \to tx + x = sx \in S_\mu$, since S_μ is closed. We will now show that S_μ *is a semigroup.* Let $x, y \in S_\mu$. Choose, as before, k_n's such that $t_n \equiv r^{n/\alpha} k_n \to 1$. Since $y \in [\bigcup_{n \geq j} S_{\mu_n}]^{-}$, for each $j = 1, 2, \ldots$, (from (3.3)), we can define, as above, a net $\{y_{(W,\, n)}\}$ such that $y_{(W,\, n)} \in W \cap S_{\mu_n}$ and $y_{(W,\, n)} \to y$. Also, if $t_{(W,\, n)} = t_n$, then, as before, $\{t_{(W,\, n)}\}$ is a subnet of $\{t_n\}$. Now $t_{(W,\, n)}x + y_{(W,\, n)} \in S_\mu$ (by (3.2)); hence, since $t_{(W,\, n)}x + y_{(W,\, n)} \to x + y$ and S_μ is closed, $x + y \in S_\mu$.

Proof of Theorem 3.2 (ii): Again we write $\mu = \mu_0 * \delta_{x_0}$ with μ_0 strictly r-semistable p.m. of index $\alpha \in (0, 1)$ [5], and show that S_{μ_0} is a convex cone. First we show that S_{μ_0} is a semigroup. Let B be a Banach space and g a continuous linear map from E to B. Let $\lambda = \mu_0 \circ g^{-1}$, then we assert that λ is strictly r-semistable with the same index α. To see this one first notes that λ is K-regular i.d. and that for any rational $s > 0$, $\lambda^s = \mu_0^s \circ g^{-1}$ (this uses the fact that the factor measure appearing in the definition of a K-regular i.d. measure on a LCTVS is unique). Then using continuity of the semigroup, one obtains that $\lambda^s = \mu_0^s \circ g^{-1}$, for all reals $s > 0$. Hence

$\lambda^{r^n} = \mu_0^{r^n} \circ g^{-1} = T_{r^{n/\alpha}}\, \mu_0 \circ g^{-1} = T_{r^{n/\alpha}}\lambda$, showing λ is strictly r-semistable of index α. Now using the fact that S_{μ_0} is the projective limit of supports of measures of the type $\mu_0 \circ g^{-1}$ (see [13]), it will follow that S_{μ_0} is a semigroup, if we can show that S_λ is a semigroup.

From Lemma 3.7, $\hat{\lambda}(f) = \exp\{\int_B (e^{if(x)} - 1)dF_\lambda(x)\}$, $f \in B^*$, where B^* is the topological dual of B . Let $\nu = \lambda * \delta_a$, where $a = \int_K x\, dF(x)$ (note that since, by Lemma 3.7, $\int_K p_K\, dF_\lambda < \infty$, $a \in B$; here K and p_K are as in Theorem 3.1). Let U_n denote the closed unit disc around θ in B of radius $1/n$, $n = 1, 2, \ldots$; we will show $\delta_{a_n} \equiv \delta_{a_{U_n}} \xrightarrow{w} \delta_a$, where $\delta_{a_{U_n}}$ is as defined in Theorem 3.1(i) . Since we already know that $\{\delta_{a_{U_n}}\}$ is tight (Theorem 3.1(ii)) , to prove $\delta_{a_n} \xrightarrow{w} \delta_a$, it is sufficient to prove that $\delta_{a_n}(f) \longrightarrow \delta_a(f)$, for every $f \in B^*$. But this follows from $|e^{if(a_n)} - e^{if(a)}| \le |\int_{K\cap U_n} f(x)\, dF_\lambda(x)| \le p_{K^0}(f) \int_{K\cap U_n} p_K\, dF_\lambda$, for every $f \in B^*$ and the dominated convergence theorem. Thus, since $S_\lambda = \overline{G(F_\lambda)} + a$ (Theorem 3.1(ii)) $= S_\lambda + a$, we have $S_\lambda = \overline{G(F_\lambda)}$. Showing S_λ is a semigroup, and hence S_{μ_0} is a semigroup. Now we will show that $S_{\mu_0^t} = S_{\mu_0}$, for $t > 0$. Let F be the Lévy measure of μ_0 ; then, by Lemma 3.7, $\mu_0(f) = \exp\{\int_E (e^{if(x)} - 1)dF(x)\}$. Therefore, letting g as above,

$$\hat{\lambda}(f) = \exp\{\int_E (e^{if(g(x))} - 1)dF(x)\} = \exp\{\int_{\{g\neq 0\}} (e^{if(g(x))} - 1)dF(x)\}$$

$$= \exp\{\int_{B\setminus\{\theta\}} (e^{if(x)} - 1)Fog^{-1}(dx)\} = \exp\{\int_B (e^{if(x)} - 1)dG(x)\},$$

for $f \in B^*$, where $G = Fog^{-1}/B\setminus\{\theta\}$. This, the fact that G is Levy (this can be proved directly by just using the definition of a Lévy measure), and the uniqueness of Lévy measure, imply that $G = F_\lambda$. Thus $\lambda^t = \exp\{\int_B (e^{if(x)} - 1)tF_\lambda\}$ (see Section 2(ii)) ; therefore $S_{\lambda^t} = \overline{G(tF_\lambda)} = \overline{G(F_\lambda)}$. Hence, since $S_{\mu_0^t}$ is the projective limit of supports of measures of the type λ^t [13], we have $S_{\mu_0^t} = S_{\mu_0}$. To finish the

proof we need only show that $sS_{\mu_0} \subseteq S_{\mu_0}$, for $0 < s < 1$. This we do in the following:

For $s \in (0, 1)$, choose by Lemma 3.4, $k_n \in \{1, \ldots, [\frac{1}{r^n}]\}$ such that $r^{n/\alpha} k_n \longrightarrow s$, as $n \longrightarrow \infty$. Now by using the facts $\mu_0^{r^n k_n} = T_{r^{n/\alpha}} \mu_0^{* k_n}$ and $S_{\mu_0^t} = S_{\mu_0}$, $t > 0$, we get

$$S_{\mu_0} = r^{n/\alpha} \overline{[S_{\mu_0}^{(k_n)}]} ,$$

where $S_{\mu_0}^{(k_n)}$ is the k_n-fold sum of S_{μ_0}. Hence for $x \in S_{\mu_0}$, $r^{n/\alpha} k_n x \in S_{\mu_0}$, so $sx \in S_\mu$, since $r^{n/\alpha} k_n x \longrightarrow sx$, as $n \longrightarrow \infty$.

<u>Proof of Theorem 3.2(iii)</u>: Since μ is symmetric and i.d., S_μ is a subgroup, by Theorem 3.1. Now, $\mu^{r^n} * \mu^{1-r^n} = \mu$ and the fact that μ^t is symmetric i.d. imply that

$$\overline{[r^n S_\mu + S_{\mu^{1-r^n}}]} = S_\mu ,$$

and $\theta \in S_{\mu^{1-r^n}}$. Consequently, $r^n S_\mu \subseteq S_\mu$, for all $n = 1, 2, \ldots$, and hence S_μ is a subspace.

<u>Remark 3.8</u>: The fact that S_{μ_0} is a subgroup and that $S_{\mu_0^t} = S_{\mu_0}$ shown above in the proof of part (ii) can also be recovered from [16]. But in order to keep the paper self contained we relied on our result rather than using [16].

4. SEMINORM INTEGRABILITY THEOREM FOR r-SEMISTABLE MEASURES

As we noted in the introduction, the proof of the result of this section is classical (see, for example, [3]); therefore, we will only give an outline of the proof and refer the reader to [11] for details, where a similar result

is obtained in Hilbert spaces.

Theorem 4.1: Let μ be a K-regular r-semistable p.m. of index $\alpha \in (0, 2)$ on E and let p be a continuous seminorm on E. Then

$$\int_E p^\delta(x)\,\mu(dx) < \infty , \tag{4.1}$$

if $\delta < \alpha$.

Outline of the Proof: Let $\nu = \mu * \bar{\mu}$, $(\bar{\mu} \equiv T_{-1}\mu)$, the symmetrization of μ. By Fubini's theorem, it is sufficient to prove (4.1) for ν. Using some arguments of the proof of Theorem 3.2(ii), we note that ν is (K-regular symmetric) r-semistable of the same index α. Let N be the quotient space $E/p^{-1}(\theta)$; if $\dot{x} = x + p^{-1}(\theta)$, set $\|\dot{x}\| = p(x)$, then $(N, \|\cdot\|)$ is a normed space, and $\lambda \equiv \nu \circ T^{-1}$ is a symmetric K-regular r-semitstable p.m. of index α (here T is the usual quotient map). Since a K-regular p.m. on a metric space has a separable support, we can assume that there exists a separable Banach subspace B of the completion $\tilde{B}$ of $(N, \|\cdot\|)$ such that $\lambda^s(B) = 1$, for all $s > 0$, (one such B is the closure in $\tilde{B}$ of the supports of λ^s, s positive rationals). Since

$$\int_E p^\delta d(\mu * \bar{\mu}) = \int_B \|\dot{x}\|^\delta \, d\lambda ,$$

by the change of variable, we need to prove (4.1) for a symmetric r-semistable p.m. of index α defined on a separable Banach space B. This is outlined below:

According to [5], we have

$$T_{r^{n/\alpha}} \lambda^{*k_n} \xrightarrow{w} \lambda ,$$

where $k_n = [\frac{1}{r^n}]$. This and Theorem 10 of [2] implies that

$$k_n \cdot T_{r^{n/\alpha}} \lambda \xrightarrow{w} F ,$$

on complements of nbds of θ in B, where F is the Lévy measure of λ. Now repeating the proof of Theorem 3.4 of [11], for given $\varepsilon > 0$ and positive integer m, one can choose t_0 such that if $t \geq t_0$, then

$$\frac{b^{m\alpha}}{a}(1+\varepsilon)^{-1} \leq \frac{Q_\lambda(t)}{Q_\lambda(b^m t)} \leq a\, b^{m\alpha}(1+\varepsilon) , \tag{4.2}$$

where $a = 1/r$, $b = r^{1/\alpha}$ and $Q_\lambda(t) = \lambda\{\dot{x} \in B: \|\dot{x}\| \geq t\}$. Now using (4.2) and following the proof of Theorem 3.5 of [11], one obtains $\int_B \|\dot{x}\|^\delta \, d\lambda < \infty$; which completes the proof.

Remark 4.2: It is worth noting that this theorem also provides a third proof of the seminorm integrability result for stable p. measures, which is different from the first two (obtained by de Acosta [1] and Kanter [8]), as long as the measures are defined on LCTV spaces and p is a continuous seminorm.

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Note: Under the assumptions of theorem 3.1 (ii), one can indeed prove that $b = \int_K x\,dF$ belongs to E and that $a_U \to b$ (and hence $S_\mu = b + \overline{G(F)}$). This fact, which shortens, to some degree, the proof of Theorem 3.2(ii), has been pointed out to us by several readers.

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Let μ be an r-semistable k-regular probability measure of index $\alpha \in (0, 2]$ on a complete locally convex topological vector space E. It is shown that the topological support S_μ of μ is a translated convex cone if $\alpha \in (0, 1)$, and a translated truncated cone if $\alpha \in (1, 2]$. Further, if $\alpha = 1$ and μ is symmetric, then it is shown that S_μ is a vector subspace

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of E . These results subsume all earlier known results regarding the support of stable measures. Results dealing with the support of infinitely divisible and the seminorm integrability for γ-semistable measures are also obtained.

gamma